Gromov's Theorem on Groups of Polynomial Growth

Zhengbo Qu
2000011611@stu.pku.edu.cn

Contents

1. Introduction

Today's topic is about an alternative proof of Gromov's Theorem. However, I must admit I'm playing a little trick on you, because it's impossible to cover the full proof of Gromov's Theorem in just a short time. The title is more of a hook to capture your interest. In fact, most time will be used to prove *Kleiner's theorem*, which is a key step in proving the whole Gromov's theorem.

First, let me introduce some preliminaries and history.

Throughout this talk, we fix a group G that is finitely generated and a finite symmetric generating set S (that is, S is closed under inversion). For every $x \in G$, the word length $\|x\|$ is the shortest length n of a word $s_1 s_2 ... s_n$ that expresses x.

Definition 1.1:

$$\|x\| := \min\{n \in \mathbb{N}_0 |\ \exists s_1, s_2, ..., s_n \in S\ s.t. x = s_1 s_2 ... s_n\}.$$

And the growth of G can be seen as the growth of the cardinality of the ball $B(r) := \{x \in G|\ \|x\| \leq r\}$

Definition 1.2: Growth: $|B(r)|$, where $B(r) := \{x \in G|\ \|x\| \leq r\}$.

Next, we can classify the growth type of a given group:

Definition 1.3: G has polynomial growth if there exist $c > 0, d \in \mathbb{N}_0$ such that $|B(r)| \leq cr^d$ for all r.

In fact, there are a total of 3 cases here, the other two are exponential and intermediate, but in this talk, we mainly focus on the case of polynomial growth. Here is a simple example:

Example: When G is abelian, then $|B(r)| = r^{\text{rank}(G)}$.

The study of growth of groups is related to various topic such as Riemmannian geometry and the amenability of groups.

As we have learned in group theory, a group is nilpotent if its lower central series terminates in finitely many steps at the trivial subgroup. And we define virtually nilpotent here:

Definition 1.4: A group G is virtually nilpotent if there is a nilpotent subgroup H of G with finite index.

In 1968, Wolf showed in [1] that if a group is virtually nilpotent, then it has polynomial growth. And in 1981, Gromov proved the converse statement in [2], which is the title of this talk: Gromov's theorem on groups of polynomial growth.

Theorem 1.1: **(Gromov's Theorem, 1981)** Every finitely generated group of polynomial growth is virtually nilpotent.

Gromov's proof uses a work about the solution to *Hilbert's fifth theorem* and is quite non-elementary.

In 2010, Klenier reproves the theorem in a more analytical way[3], using method from harmonic analysis, which is much more elementary.

In the upcoming time, I will prove Kleiner's theorem, which is also the main innovation in his paper.

2. Kleiner's Theorem.

Also, we need to give some definitions first:

Definition 2.1: A function $f : G \to \mathbb{R}$ is *Lipschitz* if:

$$\sup_{g \in G, s \in S} |f(gs) - f(g)| < +\infty,$$

and is *harmonic* if:

$$f(g) = \frac{1}{|S|} \sum_{s \in S} f(gs), \forall g \in G.$$

And here is the Kleiner's theorem:

Theorem 2.1: If $|B(r)|$ is of polynomial growth. Then the vector space H^{Lip} of Lipschitz harmonic functions on G is finite dimensional.

For simlicity, we will assume a slightly stronger condition than polynomial growth, namely *bounded doubling*:

$$\exists C > 0, s.t. \ |B(2r)| \leq C|B(r)|, \forall r > 0$$

The entire proof will go in two steps.

Initially, we'll state two important inequalities: the Poincaré inequality and the reverse one. The proof of these inequalities involves a lot of techniques. So we'll first apply these two inequalities to establish the theorem, and subsequently, we'll go back to prove these two inequalities.

And here's the two inequality:

Lemma 2.1: **(Poincaré inequality)** For every function $f : G \to \mathbb{R}$, if f has mean 0 on $B(r)$, then its l^2-norm on $B(r)$ is bounded by the fluctuation on $B(2r)$:

$$\sum_{x \in B(r)} f^2(x) \leq \frac{|B(2r)|}{|B(r)|} \cdot 2r^2 \sum_{x,y \in B(2r), x \sim y} (f(x) - f(y))^2$$

Lemma 2.2: **(Reverse Poincaré inequality)** For every harmonic function $f : G \to \mathbb{R}$, then its fluctuation on $B(r)$ is bounded by the l^2-norm on $B(2r)$:

$$\sum_{x,y \in B(r), x \sim y} (f(x) - f(y))^2 \leq |S| \cdot \frac{4}{r^2} \sum_{x \in B(2r)} f^2(x)$$

2.1. the proof of Kleiner's theorem

Proof: First assume that $\dim(H^{\text{Lip}}) \geq n$, where n will be determined later. Denote by V the n-dimensional subspace of H^{Lip} ($V \subseteq H^{\text{Lip}}, \dim(V) = n$).

Let k be a natural number to be determined soon, and fix r for a moment.

Let $\mathcal{A}_r$ be a maximal collection of disjoint balls of radius $\frac{r}{2}$ with centers in $B(kr)$, let $\mathcal{B}_r$ be the collection of balls with the same centers of the balls in $\mathcal{A}_r$, but of radius r. *(*$\mathcal{B}_r := \{2A | A \in \mathcal{A}_r\}$*)*

Let V_r be the vector subspace of V consisting of functions in V that average on each ball in $\mathcal{B}_r$. ($V_r := \{f \in V | \forall B \in \mathcal{B}_r, f \text{ has mean } 0 \text{ on } B\}$)

Note that the co-dimension of V_r in V is at most

$$|\mathcal{B}_r| \leq \frac{|B(kr + \frac{r}{2})|}{|B(\frac{r}{2})|} = O(1) =: C,$$

where the constant C depends only on G.

For every harmonic function $f \in V_r$, we note that $B(kr) \subseteq \cup_{B \in \mathcal{B}_r} B$, and each point in $B(kr)$ is covered by $2B$ for at most $|B(2r + \frac{r}{2})| / |B(\frac{r}{2})| = O(1)$ many $B \in \mathcal{B}_r$.

Next, we apply the two inequalities.

$$\begin{aligned}\sum_{x\in B(kr)} f^2(x) &\leq \sum_{B\in\mathcal{B}_r}\sum_{x\in B} f^2(x)\\ &\lesssim r^2 \sum_{B\in\mathcal{B}_r}\sum_{x,yin2B,x\sim y} (f(x)-f(y))^2\\ &\lesssim r^2 \sum_{x,y\in B(kr+2r),x\sim y} (f(x)-f(y))^2\\ &\lesssim \frac{1}{(k+2)^2}\sum_{x\in B(2(k+2)r)} f^2(x)\end{aligned}$$

Now take k large enough, we can find a natural number d that will be determined later such that for all $f \in V_r$,

$$3^d \sum_{x\in B(kr)} f^2(x) \leq \sum_{x\in B(3kr)} f^2(x)$$

Next, consider the quadratic form Q_r defined on V defined by $Q_{r(f)} := \sum_{x\in B(r)} f^2(x)$. The above formula implies that, for all $f \in V_r$, we have:

$$3^d Q_{kr}(f) \leq Q_{3kr}(f).$$

Since the kernels of Q_r's form a descending chain of vector spaces, there exists r_0 such that Q_r is positive-definite for all $r \geq r_0$.

Now fix a basis $\{f_1, ..., f_n\}$ for V and assume that $f_{i(e)} = 0$ and is 1-Lipschitz is f_i is non-constant. Then for all $r \geq r_0$, we define:

$$E_r := \{(c_1, ..., c_n) \in \mathbb{R}^n \mid Q_r(c_1 f_1 + ... + c_n f_n) \leq 1\},$$
$$q(r) = \mathrm{vol}(E_r).$$

Note that $q(r)/q(r')$ does not depend on the choice of the basis.

Since $|B(r)|$ is at most polynomial in r, there exists $d \in \mathbb{N}$, such that:

$$\sum_{x\in B(r)} f_i^2(x) \leq r^d, \forall i = 1, ..., n.$$

By Cauchy-Schwarz inequality, we have:

$$\begin{aligned}Q_r(c_1 f_1 + ... + c_n f_n) &= \sum_{x\in B(r)} (c_1 f_1 + ... + c_n f_n)^2(x)\\ &\leq n \sum_{x\in B(r)} \left(c_1^2 f_1^2(x) + ... + c_n^2 f_n^2(x)\right)\\ &\leq n^2 r^d \sum_{i=1,...,n} c_i^2.\end{aligned}$$

Therefore, in $\mathbb{R}^n$, we have:

$$B\left(0, 1/nr^{\frac{d}{2}}\right) \subseteq E_r.$$

Which implies that

$$q(r) \geq v_n \left(n r^{\frac{d}{2}}\right)^{-n},$$

where v_n is a constant.

The remaining part is just linear algebra. For every Q_r, we have a symmetric matrix:

$$M(Q_r) = \left(Q_r(f_i, f_j)\right)_{i,j}.$$

And we define the determinant of Q_r as $\det(Q_r) := \det(M(Q_r))$.

After linear transformation, we can assume that $M(Q_{kr}), M(Q_{3kr})$ are of the form:

$$\begin{pmatrix} A_1 & B_1 \\ B_1^T & C_1 \end{pmatrix}, \begin{pmatrix} A_3 & \\ & C_3 \end{pmatrix},$$

Where A_1, A_3 act on V_r. And we compute that:

$$\begin{aligned} \frac{q_{3kr}}{q_{kr}} &= \frac{\det(Q_{kr})}{\det(Q_{3kr})} \\ &= \frac{\det(A_1)\det(C_1 - B_1^T A_1^{-1} B_1)}{\det(A_3)\det(C_3)} \\ &\leq \frac{\det(A_1)\det(C_1)}{\det(A_3)\det(C_3)}. \end{aligned}$$

Since $3^d Q_{kr}(f) \leq Q_{3kr}(f)$, for all $f \in V_r$, we have $Q_{kr}(f) \leq Q_{3kr}(f)$ for all $f \in V$. Thus, we have $3^d A_1 \preceq A_3$ and $C_1 \preceq C_3$, which implies that:

$$\frac{q_{kr}}{q_{3kr}} \geq \left(3^d\right)^{\dim(V_r)} \geq 3^{d(n-c)}.$$

Now we can choose n large enough such that $3^{d(n-c)} \geq 2^{nd}$, hence we get:

$$\begin{aligned} q(kr) &\geq 2^{nd} q(3kr) \\ &\geq 2^{mnd} q(3^m kr) \\ &\geq 2^{mnd} v_n \cdot \left(n(3^m kr)^{\frac{d}{2}}\right)^{-n} \\ &\geq C'(n,d) \cdot \left(2/\sqrt{3}\right)^{mnd}, \end{aligned}$$

which is a contradiction. □

2.2. Proof of the Poincaré inequality

Proof:

$$\text{LHS} = \frac{1}{2|B(r)|} \sum_{x,y \in B(r)} (f(x) - f(y))^2.$$

Then for all $z \in B(2r)$, fix a shortest path $e = z_0, z_1, \ldots, z_{\|z\|} = z$ from e to z.

Given $x, y \in B(r)$, let $z = x^{-1}y \in B(2r)$, then we have:

$$f(x) - f(y) = \sum_{i=1}^{\{\|z\|\}} (f(xz_{i-1}) - f(xz_i)),$$

applying the Cauchy-Schwarz inequality, we have:

$$(f(x) - f(y))^2 \le \|z\| \sum_{i=1}^{\|z\|} (f(xz_{i-1}) - f(xz_i))^2.$$

Summing over all $x, y \in B(r)$, we get:

$$\begin{aligned} 2|B(r)| \cdot \text{LHS} &\le \sum_{x,y \in B(r), z = x^{-1}y} \|z\| \sum_{i=1}^{\|z\|} (f(xz_{i-1}) - f(xz_i))^2 \\ &= \sum_{z \in B(2r)} \|z\| \left(\sum_{x \in B(r), xz \in B(r)}^{\|z\|} (f(xz_{i-1}) - f(xz_i))^2 \right). \end{aligned}$$

Fix z and i for a moment, we can find that $x_{z(i-1)}, xz_i \in B(2r)$, and the directed edge $(xz_{i-1}, xz_i) \ne (x' z_{i-1}, x' z_i)$ if $x \ne x'$. Therefore, we have:

$$\sum_{x \in B(r), xz \in B(r)}^{\|z\|} (f(xz_{i-1}) - f(xz_i))^2 \le \sum_{x,y \in B(2r), x \sim y} (f(x) - f(y))^2.$$

Then we get:

$$2|B(r)| \cdot \text{LHS} \le \sum_{z \in B(2r)} \|z\|^2 \sum_{x,y \in B(2r), x \sim y} (f(x) - f(y))^2 \le |B(2r)| \cdot 4r^2 \sum_{x,y \in B(2r), x \sim y} (f(x) - f(y))^2.$$

□

2.3. Proof of the reverse Poincaré inequality

We first introduce some notations:

Definition 2.3.1: Given $f : G \to \mathbb{R}$, $s \in S$, let

$$f_s(x) := f(xs), \partial_s f := f_s - f$$

It is easy to check that:

Proposition 2.3.1:
1. $\sum_{s \in S} \partial_{s^{-1}} \partial_s f = 0$ if f is harmonic.
2. $\sum_{x \in G} f(x) \partial_s g(x) = \sum_{x \in G} \partial_{s^{-1}} f(x) g(x)$ when f or g is finitely supported.

Now we can prove the reverse Poincaré inequality.

Proof: Fix a harmonic function f, let $\varphi : G \to [0, 1]$ be as follows:

$$\varphi(x) = \begin{cases} 1, \|x\| \leq r \\ 2 - \frac{\|x\|}{r}, r < \|x\| < 2r. \\ 0, 2r \leq \|x\| \end{cases}$$

Note that, for all $s \in S$, we have:

$$\partial_s(f\varphi^2) = (\partial_s f)\varphi^2 + f_s(\partial_s \varphi^2)$$
$$\partial_s \varphi^2 = (\partial_s \varphi)(2\varphi + \partial_s \varphi)$$

So we have:

$$(\partial_s f)(\partial_s(f\varphi^2)) = (\partial_s f)^2\varphi^2 + f_s(\partial_s f)(\partial_s \varphi)(2\varphi + \partial_s \varphi),$$

then

$$\begin{aligned}\frac{1}{2}(\partial_s f)^2\varphi^2 + 2f_s(\partial_s f)(\partial_s \varphi)\varphi &\geq -2f_s^2(\partial_s \varphi)^2 \\ &\geq \frac{1}{2}(\partial_s f)^2\varphi^2 - f_{s(f_s+f)}(\partial_s \varphi)^2 \\ &\geq \frac{1}{2}(\partial_s f)^2\varphi^2 - \frac{1}{2}(3f_s^2 + f^2)(\partial_s \varphi)^2.\end{aligned}$$

Summing over all $s \in S$ and all $x \in G$, we get:

$$\begin{aligned}\sum_{s\in S}\sum_{x\in G} \partial_s f \cdot \partial_s(f\varphi^2) &= \sum_{s\in S}\sum_{x\in G}(\partial_{s^{-1}}\partial_s f)(f\varphi^2) \\ &= \sum_{x\in G}\left(\sum_{s\in S}\partial_{s^{-1}}\partial_s f\right)(f\varphi^2) \\ &= 0.\end{aligned}$$

Which implies that:

$$\sum_{s\in S}\sum_{x\in G}(\partial_s f(x))^2\varphi(x)^2 \leq \sum_{s\in S}\sum_{x\in G}(3f(xs)^2 + f(x)^2)(\partial_s \varphi(x))^2$$

And we have:

$$\text{LHS} \geq \sum_{x,y\in B(r), x\sim y}(f(x) - f(y))^2,$$
$$\text{RHS} \leq |S| \cdot \frac{4}{r^2}\sum_{r\leq\|x\|\leq 2r} f^2(x),$$

which completes the proof.

□

Bibliography

[1] J. A. Wolf, "Growth of finitely generated solvable groups and curvature of Riemannian manifolds," *Journal of Differential Geometry*, vol. 2, no. 4, pp. 421–446, 1968, doi: 10.4310/jdg/1214428658.

[2] M. Gromov, "Groups of polynomial growth and expanding maps," *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, vol. 53, no. 1, pp. 53–78, Dec. 1981, doi: 10.1007/BF02698687.

[3] B. Kleiner, "A new proof of Gromov's theorem on groups of polynomial growth," *Journal of the American Mathematical Society*, vol. 23, no. 3, pp. 815–829, Dec. 2009, doi: 10.1090/S0894-0347-09-00658-4.