Dehn-Nielsen-Baer theorem and Semiconjugacy

Zhengbo Qu
2000011611@stu.pku.edu.cn

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In this seminar, we are going to give a talk on "Rigidity of mapping class group actions on S^1" [1]. And my task is to go through Dehn-Nielsen-Baer theorem and some notions about semiconjugacy, which aims at providing some background knowledge for the main theorem in this paper

1. The Dehn-Nielsen-Baer theorem

Remark: We are mainly follow the expositions from [2, Chapter 8].

The Dehn-Nielsen-Baer theorem is a fundamental result in the theory of mapping class groups, it relates a topological object, the mapping class group, to an algebraic object, the outer automorphism group of the fundamental group of a surface. Specifically, it says that $\mathrm{Mod}(S_g)$ is isomorphic to an index two subgroup of $\mathrm{Out}(\pi_1(S_g))$.

We first introduce some basic concepts.

1.1. Preliminaries

Definition 1.1.1: Let S be a surface without boundary, the **extended mapping class group** of S, denoted by $\mathrm{Mod}^{\pm}(S)$, is the group of isotopy classes of homeomorphisms of S, including the orientation-reversing ones.

And we have the following split short exact sequence:

$$1 \to \mathrm{Mod}(S) \to \mathrm{Mod}^{\pm}(S) \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

Example: (Cf. [2, Chapter 2] for more details)

1. $\mathrm{Mod}^{\pm}(S^2) = \mathbb{Z}/2\mathbb{Z}$.
2. $\mathrm{Mod}(T^2) = \mathrm{SL}(2, \mathbb{Z})$, and $\mathrm{Mod}^{\pm}(T^2) = \mathrm{GL}(2, \mathbb{Z})$.
3. $\mathrm{Mod}(S_{0,3}) = \Sigma_3 \times \mathbb{Z}/2\mathbb{Z}$.

Definition 1.1.2: For a group G, let $\mathrm{Aut}(G)$ denote the group of group automorphisms of G, and $\mathrm{Inn}(G)$ denote the group of inner automorphisms of G. Then the **outer automorphism group** of G is defined as $\mathrm{Out}(G) = \mathrm{Aut}(G)/\ \mathrm{Inn}(G)$.

Proposition 1.1.1: (Cf. [2, Theorem 2.5] or [3, Proposition 1B.9]) Let S be a surface with $\chi(S) \leq 0$, then S is a $K(\pi_1(S), 1)$ space. We have the following bijective correspondence:

$$\{\text{Free homotopy classes of maps } S \to S\}$$
$$\updownarrow$$
$$\{\text{Conjugacy classes of homomorphisms } \pi_1(S) \to \pi_1(S)\}.$$

If $\phi : S \to S$ is a homeomorphism, then the induced homomorphism $\phi_* : \pi_1(S) \to \pi_1(S)$ is an automorphism of $\pi_1(S)$. Then we have an well-defined map by the above correspondence:

$$\sigma : \mathrm{Mod}^{\pm}(S) \to \mathrm{Out}(\pi_1(S)).$$

It is not hard to see that σ is injective, and we have the following theorem.

Theorem 1.1.1: (**Dehn-Nielsen-Baer Theorem**) Let $g \geq 1$. The map

$$\sigma : \mathrm{Mod}^{\pm}(S_g) \to \mathrm{Out}(\pi_1(S_g))$$

is an isomorphism.

Remark: In [2], the main proof of this theorem uses quasi-isometry methods [2, Section 8.2], but this proof is relatively complex. Due to time constraints in the seminar, we will use an alternative proof method using pants decomposition, following the approach in [2, Section 8.3].

Note that in the following proof, we need to assume $g \geq 2$, but this assumption is harmless, since when $g = 1$, we already have the fact that $\mathrm{Mod}^{\pm}(T^2) = \mathrm{GL}(2, \mathbb{Z})$

1.2. Proof of the Dehn-Nielsen-Baer theorem: the pants decomposition approach

Let S be a surface with $\chi(S) < 0$. Since S is a $K(\pi_1(S), 1)$ space, every outer automorphism of $\pi_1(S)$ is induced by some map $S \to S$. Since $\pi_i(S) = 0$ for $i > 1$, this self map actually induces isomorphism on all homotopy groups, and hence is a homotopy equivalence by the Whitehead theorem(cf. [3, Theorem 4.5]).

Thus, for the surjectivity part of Theorem 1.1.1, it suffices to show that every homotopy equivalence of S is homotopic to a homeomorphism.

Theorem 1.2.1: If $g \geq 2$, then every homotopy equivalence $\phi : S_g \to S_g$ is homotopic to a homeomorphism.

The remaining part of this subsection is to prove Theorem 1.2.1. We will first introduce the concept of pants decomposition.

Definition 1.2.1: A **pair of pants** is a compact surface of genus 0 with 3 boundary components. Let S be a compact surface with $\chi(S) < 0$. A **pants decomposition** of S is a collection of pairwise disjoint simple closed curves on S such that S minus the curves is a disjoint union of pairs of pants.

Remark: Equivilantly, a pants decomposition is a maximal collection of disjoint, essential simple closed curves on S such that no two of them are isotopic. It is not hard to see that these two definitions are equvivalent.

A pair of pants has Euler characteristic -1. Since cutting a surface along a simple closed curve preserves the Euler characteristic, we have the following lemma.

Lemma 1.2.1:
1. A pants decomposition of S cuts S into $-\chi(S)$ pairs of pants.
2. A pants decomposition of S has $\frac{-3\chi(S)-b}{2} = 3g + b - 3$ curves.
3. In particular, a pants decomposition of S_g has $3g - 3$ curves and $2g - 2$ pairs of pants.

To prove Theorem 1.2.1, we need the following lemma.

Lemma 1.2.2: If $\varphi : R \to R'$ is a continuous map such that $\varphi|_{\partial R}$ is a homeomorphism, then there is a homotopy of φ to a homeomorphism $R \to R'$, such that the homotopy restricts to the identity on ∂R.

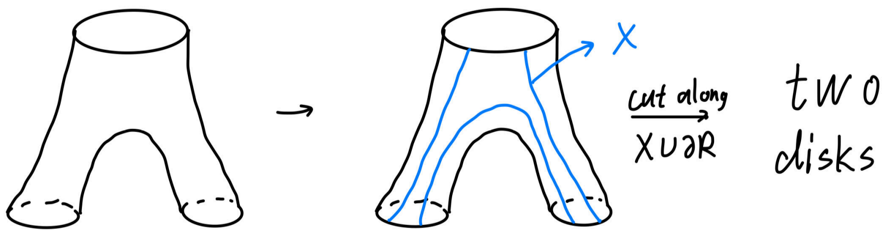

Figure 1: A pair of pants cut along $X \cup \partial R$.

Proof: Let X be the union of three disjoint arcs in R', one connecting each pair of boundary components of R'. Then $R' - (\partial R \cup X)$ is homeomorphic to a disjoint union of two open disks (see Figure 1).

We may assume φ to be smooth, and so $\varphi^{-1}(X)$ is a properly embedded 1-manifold in R with boundary lying in ∂R. If any component of $\varphi^{-1}(X)$ is a closed curve, then it is nullhomotopic since it is not homotopic to a boundary component of R, and we can modify φ to remove this component. As a result, we can assume $\varphi^{-1}(X)$ has exactly three arcs.

Since $\varphi|_{\partial R}$ is a homeomorphism, and it takes distinct boundary components to distinct boundary components, we can assume that $\varphi^{-1}(X)$ consists of three arcs connecting the boundary components of R. We can modify φ such that it restricts to a homeomorphism on each component of X. By the Alexander lemma (cf.[2, Lemma 2.1]), we have that φ is homotopic to a homeomorphism. □

With this theorem, we can prove the main theorem.

Proof: (of Theorem 1.2.1)

We modify the homotopy equivalence $\phi : S_g \to S_g$ step by step until it is a homeomorphism, at each stage, the resulting map will be called ϕ. Choose a pants decomposition $\mathcal{P}$ of S_g consisting of smooth simple closed curves.

1. (**Transversality**) We first approximate ϕ by a smooth map that is transverse to $\mathcal{P}$, and we can actually make it close enough to the original ϕ. Thus we can assume this smooth map is homotopic to the original ϕ.
   By transversality, we have $\phi^{-1}(\mathcal{P})$ is a collection of simple closed curves. In fact, we can assume that these curves are essential, since an inessential curve bounds a disk, and we can homotope ϕ using the disk to remove the curve.
   To conclude, ϕ is a smooth map that is transverse to $\mathcal{P}$, and $\phi^{-1}(\mathcal{P})$ is a collection of essential simple closed curves.
2. (**Homeomorphism on $\boldsymbol{\phi^{-1}(\mathcal{P})}$**) Since ϕ_* is an automorphism on $\pi_1(S_g)$, it take primitive conjugacy classes to primitive conjugacy classes. Thus, the restriction of ϕ

to any componet of $\phi^{-1}(\mathcal{P})$ has degree ± 1 as a map $S^1 \to S^1$. Therefore, we can homotope ϕ such that its restriction on each componet of $\phi^{-1}(\mathcal{P})$ is a homeomorphism.

3. (**Reducing components in $\boldsymbol{\phi^{-1}(\mathcal{P})}$**) Since ϕ is a homotopy equivalence, it has degree ± 1, and it is surjective. Thus, $\phi^{-1}(\mathcal{P})$ has at least $3g-3$ components. If it had more, then there exists two components that are isotopic (see the equvivalent definition of pants decomposition), and we can homotope ϕ through the annulus between them to reduce the number of components.
   Thus, we can assume that $\phi^{-1}(\mathcal{P})$ has exactly $3g-3$ components.
4. (**Applying Lemma 1.2.2**) Now, ϕ maps each component of $S_g - \phi^{-1}(\mathcal{P})$ to a single component of $S_g - \mathcal{P}$. For each component R of $S_g - \phi^{-1}(\mathcal{P})$ and the corresponding component R' of $S_g - \mathcal{P}$, we can apply Lemma 1.2.2 to homotope ϕ on R to a homeomorphism $R \to R'$.
   After all these steps, we obtain a homeomorphism $\phi : S_g \to S_g$.

□

1.3. Some remarks

Here we present some additional knowledge about the theorem, which will be useful in the paper.

Recall that we have the following short exact sequence, i.e., the **Birman exact sequence**(cf. [2, Theorem 4.6]):

$$1 \longrightarrow \pi_1(S_g) \overset{\text{push}}{\longrightarrow} \mathrm{Mod}^{\pm}(S_{g,1}) \overset{\text{forget}}{\longrightarrow} \mathrm{Mod}^{\pm}(S_g) \longrightarrow 1$$

The Dehn-Nielsen-Baer theorem can be used to relate the group $\mathrm{Mod}(S_g)$ to the group $\mathrm{Mod}(S_{g,1})$, wehre $S_{g,1}$ is a surface of genus $g \geq 2$ with one marked point, actually, we have the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Inn}(\pi_1(S_g)) \arrow[r] \arrow[d, leftrightarrow, "\approx"] & \mathrm{Aut}(\pi_1(S_g)) \arrow[r] \arrow[d, leftrightarrow, "\approx"] & \mathrm{Out}(\pi_1(S_g)) \arrow[r] \arrow[d, leftrightarrow, "\approx"] & 1 \\
1 \arrow[r] & \pi_1(S_g) \arrow[r] & \mathrm{Mod}^{\pm}(S_{g,1}) \arrow[r] & \mathrm{Mod}^{\pm}(S_g) \arrow[r] & 1
\end{tikzcd}$$

The isomorphism between $\mathrm{Inn}(\pi_1(S_g))$ and $\pi_1(S_g)$ is because $\pi_1(S_g)$ has trivial center, and the isomorphism between $\mathrm{Out}(\pi_1(S_g))$ and $\mathrm{Mod}^{\pm}(S_g)$ is because of the Dehn-Nielsen-Baer theorem, and the central vertical isomorphism comes from the five lemma.

2. Standard action and semiconjugacy

Here we introduce some basic concepts about standard action and semiconjugacy, mainly following the Introduction section of [1].

2.1. Standard action

By uniformization theorem, fixing a hyperbolic metric on S_g, the universal cover $\widetilde{S}_g$ can be identified with the hyperbolic plane $\mathbb{H}^2$, which has a natural compactification to a closed disc.

For $f \in \mathrm{Homeo}(S_g)$ that fix the marked point, let $x \in \mathbb{H}^2$ be a lift of the marked point of S_g. Then f has a unique lift $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^2$ that fixes x. One can show that the actions of $\tilde{f}$ on $\mathbb{H}^2$ extends to a homeomorphism on the boundary S^1, which depends only on the isotopy class of f.

Thus, this gives a well-defined homomorphism $\mathrm{Mod}^{\pm}(S_{g,1}) \cong \mathrm{Aut}(\pi_1(S_g)) \to \mathrm{Homeo}(S^1)$, And we call this action the **standard action** of $\mathrm{Aut}(\pi_1(S_g))$ on S^1.

2.2. Semiconjugacy

Since ANY C^0 action of an infinite group on S^1 can be modified using Denjoy trick to produce nonconjugate examples, we need to focus on semeconjugacy, and here is the detailed definition.

Let $\mathrm{Aut}_+(\pi_1(S_g))$ be subgroup of $\mathrm{Aut}(\pi_1(S_g))$ that corresponds to to $\mathrm{Mod}(S_{g,1})$

Definition 2.2.1: Two actions $\rho_1, \rho_2 : \Gamma \to \mathrm{Homeo}_+(S^1)$ are said to be **Semiconjugate** if there exists an equivariant, cyclic order preserving bijection from some orbit of S^{11} under ρ_1 to some orbit of S^1 under ρ_2.

2.3. Main theorem of the paper [1]

And we are finally ready to state the main theorem of the paper.

Theorem 2.3.1: Let $\rho : \mathrm{Aut}_+(\pi_1(S_g)) \to \mathrm{Homeo}_+(S^1)$ be a homomorphism. Up to reversing the orientation of the circle, we have the following.

1. If $g = 2$, then ρ is either conjugate to a subgroup of $\mathbb{Z}/10\mathbb{Z}$ acting by rotations, or is semiconjugate to the standard action.
2. If $g \geq 3$, then ρ is either trivial or semiconjugate to the standard action.

Bibliography

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