On the Iterated Monodromy Groups of Expanding Thurston Maps 🏫

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1. Introduction

1.1 Expanding Thurston maps

Let S^2 denote a topological 2-sphere, $f : S^2 \to S^2$ be a branched covering map. We denote by $\operatorname{crit} f$ the set of critical points of f.

A point $y \in S^2$ is a *postcritical point* of f if $y = f^n(x)$ for some $x \in \operatorname{crit} f$ and $n \in \mathbb{N}$. The set of postcritical points of f is denoted by $\operatorname{post} f$. Note that $\operatorname{post} f = \operatorname{post} f^n$ for all $n \in \mathbb{N}$.

Definition 1 (Thurston maps). A Thurston map is a branched covering map $f : S^2 \to S^2$ on S^2 with $\deg f \geq 2$ and $\operatorname{card} (\operatorname{post} f) < +\infty$.

1.1 Expanding Thurston maps

An important properties of Thurston maps is that they have cell decompositions. We record the following definitions from [1].

Definition 2 (Cell decompositions). Let $\boldsymbol{D}$ be a collection of cells in S^2. We say that $\boldsymbol{D}$ is a *cell decomposition of* S^2 if the following conditions are satisfied:

- the union of all cells in $\boldsymbol{D}$ is equal to S^2,
- if $c \in \boldsymbol{D}$, then ∂c is a union of cells in $\boldsymbol{D}$,
- for $c_1, c_2 \in \boldsymbol{D}$ with $c_1 \neq c_2$, we have $\operatorname{inte}(c_1) \cap \operatorname{inte}(c_2) = \emptyset$,
- every point in S^2 has a neighborhood that meets only finitely-many cells in $\boldsymbol{D}$.

1.1 Expanding Thurston maps

Proposition 3 Let $f : S^2 \to S^2$ be a Thurston map, $\mathcal{C} \subseteq S^2$ be a Jordan curve with post $f \subseteq \mathcal{C}$. Then there exists a unique sequence of cell decompositions $\boldsymbol{D}^n = \boldsymbol{D}^n(f, \mathcal{C})$ for $(f, \mathcal{C})$.

Thus, we can define the cell decomposition for arbitrary Thurston maps, and we have tiles, edges, and vertices in the cell decompositions.

Definition 4 (Tiles). Given a cell decomposition $\boldsymbol{D}(f, \mathcal{C})$, we say that a cell of dimension 2 in $\boldsymbol{D}^n$ a n-*tile*, and similarly, we say that a cell of dimension 1 in $\boldsymbol{D}^n$ a n-*edge* and a cell of dimension 0 in $\boldsymbol{D}^n$ a n-*vertex*.

1.1 Expanding Thurston maps

We can now give a definition of expanding Thurston maps.

Definition 5 (Expansion). A Thurston map $f : S^2 \to S^2$ is called *expanding* if there exists a metric d on S^2 that induces the standard topology on S^2 and a Jordan curve $\mathcal{C} \subseteq S^2$ containing post f such that

$$\lim_{n \to +\infty} \max\{\operatorname{diam}_d(X) : X \in \boldsymbol{X}^n(f, \mathcal{C})\} = 0.$$

1.2 Solenoids and leaves

We sometimes need to study the dynamics of expanding Thurston maps in a more general setting. In this case, we introduce the notion of solenoids.

Definition 6. Given an expanding Thurston map $f : S^2 \to S^2$, we define the *solenoid* of f to be

$$\mathcal{S}(f) := \left\{ (x_n)_{n\in\mathbb{N}} \in \left(S^2\right)^{\mathbb{N}_0} : f(x_{n+1}) = x_n \quad \text{for all} \quad n \in \mathbb{N}_0 \right\}.$$

The solenoid $\mathcal{S}(f)$, as a subset of $\left(S^2\right)^{\mathbb{N}_0}$, inherits the product topology from the product topology on $\left(S^2\right)^{\mathbb{N}_0}$. Moreover, it is the inverse limit of

$$\ldots \xrightarrow{f} S^2 \xrightarrow{f} S^2 \xrightarrow{f} S^2 \xrightarrow{f} S^2.$$

1.2 Solenoids and leaves

The solenoid $\mathcal{S}(f)$ has wild topology, actually, it is connected but not path-connected. In order to study its properties, we need to introduce the notion of **leaves** of solenoids, which are path-connected of the solenoid. We have the following characterization of leaves.

Proposition 7. *Two points* $(x_n)_{n\in\mathbb{N}_0}$ *and* $(y_n)_{n\in\mathbb{N}_0}$ *in* $\mathcal{S}(f)$ *are in the same leaf in* $\mathcal{S}(f)$ *if and only if* $\sup_{n\in\mathbb{N}_0}\Lambda^n\rho(x_n, y_n) < +\infty$.

Note that we can lift tiles of S^2 to leaves on $\mathcal{S}(f)$ and talk about tiles and adjacency graphs on leaves.

1.3 Iterated monodromy groups

Choose a point $t \in S^2 \setminus \operatorname{post} f$, and consider the rooted *tree of preimages* with the vertex set

$$T_f = \bigsqcup_{n \geq 0} f^{-n}(t),$$

where a vertex $z \in f^{-(n+1)}(t)$ is connected to the vertex $p(z) \in f^{-n}(t)$ by an edge. The point t is the *root* of the tree T_f.

And we can define the *iterated monodromy action* and the *iterated monodromy group* as follows.

1.3 Iterated monodromy groups

Definition 8. Let $\gamma \in \pi_1(S^2 \setminus \operatorname{post} f, t)$ be a loop starting and ending in t, then for every $n \in \mathbb{N}_0$ and $z \in f^{-n}(t)$, there exists precisely one f^n -lift $\gamma_z = f^{-n}(\gamma)[z]$ of γ starting at z. Let $\gamma(z)$ be the end of γ_z. Then the map $\gamma : z \mapsto \gamma(z)$ is a permutation of the level $f^{-n}(t)$ of the tree T_f, and it is called the *monodromy action* of γ on $f^{-n}(t)$.

The permutation $z \mapsto \gamma(z)$ is an automorphism of the tree T_f.

Thus we get an action of $\pi_1(S^2 \setminus \operatorname{post} f, t)$ on the tree T_p. And we call this action the *iterated monodromy action* of $\pi_1(S^2 \setminus \operatorname{post} f, t)$ on the tree T_f.

1.3 Iterated monodromy groups

The quotient of the iterated monodromy action of $\pi_1(S^2 \setminus \operatorname{post} f, t)$ by the kernel of the action is called the *iterated monodromy group* of f and is denoted by $\operatorname{IMG}(f)$.

In order to better understand the iterated monodromy group, we introduce the notion of *Schreier graph*.

Definition 9. Let G be a group generated by a finite set S and G acts on a set X. Then the corresponding *Schreier graph* $\Gamma(G, S, X)$ is a graph with vertex se X and edge set $\{(x, s \cdot x) : x \in X, s \in S\}$.

1.3 Iterated monodromy groups

Clearly, we can consider the iterated monodromy group as a group acting on the tree T_f, and thus we can define the Schreier graph of the iterated monodromy group.

In particular, suppose that $\mathrm{IMG}\ (f)$ is generated by a finite set S, the n-th level T_f^n of the tree T_f is invariant under the action of $\mathrm{IMG}\ (f)$, and we denote the associated Schreier graph by $\Gamma_n = \Gamma\left(\mathrm{IMG}\ (f), S, T_f^n\right)$.

Moreover, we can define the *infinite Schreier graph* $\Gamma_\infty = \Gamma\left(\mathrm{IMG}\ (f), S, \partial T_f\right)$, where ∂T_f is the boundary of the tree T_f.

1.4 Local convergence of graphs

Definition 10. A *rooted graph* is a pair (G, o), where G is a graph and o is a vertex of G, called the *root* of the rooted graph. A rooted graph is *isomorphic* to another rooted graph if there exists an isomorphism of the underlying graphs that maps the root of the first graph to the root of the second graph.

We denote by $\mathcal{G}^\bullet$ the set of isomorphism classes of locally finite connected rooted graphs. We can then define a metric on $\mathcal{G}^\bullet$ as follows.

1.4 Local convergence of graphs

Given an element $(G, o) \in \mathcal{G}^{\bullet}$, the finite rooted graph $B_G(o, r)$ is the subgraph of (G, o) induced by the set of vertices at distance at most r from o. We equip $\mathcal{G}^{\bullet}$ with a metric d_{loc} defined by

$$d_{loc}((G_1, o_1), (G_2, o_2)) := 2^{-r},$$

where r is the largest integer such that $B_{G_1}(o_1, r)$ and $B_{G_2}(o_2, r)$ are isomorphic as graphs.

Note that $G_{\bullet}$ equipped with the metric d_{loc} is a Polish space, thus we can talk about convergence in distribution of a sequence of random variables taking values in $\mathcal{G}^{\bullet}$.

1.4 Local convergence of graphs

We only consider the situation where random variable is a finite rooted random graph (G_n, o_n), such that given G_n, the root o_n is chosen uniformly at random among the vertices of G_n. It justifies the following definition.

Definition 11. Let $\{G_n\}$ be a sequence of random finite graphs. We say that G_n *converges locally* to a (possibly infinite) random rooted graph $(G, o) \in \mathcal{G}^\bullet$, and denote it by $G_n \stackrel{loc}{\rightarrow} (G, o)$, if for every $r \in \mathbb{N}$, $B_{G_n}(o_n, r)$ converges in distribution to $B_G(o, r)$ as $n \to +\infty$, where o_n is a uniformly chosen root of G_n.

1.4 Local convergence of graphs

Now we can present a seminal theorem about local convergence of graphs.

Theorem 12 (Benjamini–Schramm[2]). *Let $M < +\infty$ and let G_n be rooted random finite planar graphs with degrees bounded by M such that $G_n \overset{loc}{\to} (G, o)$. Then (G, o) is almost surely recurrent.*

This theorem is a key result in the study of local convergence of graphs, and it will be used in the proof of the main theorem in this paper.

2. What we have done

2.1 Main results

Theorem 12. *Let $f : S^2 \to S^2$ be an expanding Thurston map without periodic critical points. Then the infinite Schreier graph $\Gamma_\infty = \Gamma\left(\mathrm{IMG}\,(f), S, \partial T_f\right)$ of the iterated monodromy group $\mathrm{IMG}\,(f)$ is almost surely recurrent.*

2.2 Sketch of the proof

Step 1: Translate the n-th Schreier graphs Γ_n to the adjacency graphs $\mathcal{G}_n$ of n-tiles on S^2, and the infinite Schreier graph Γ_∞ to the adjacency graphs $\mathcal{G}_\infty$ of tiles on the solenoid $\mathcal{S}(f)$.

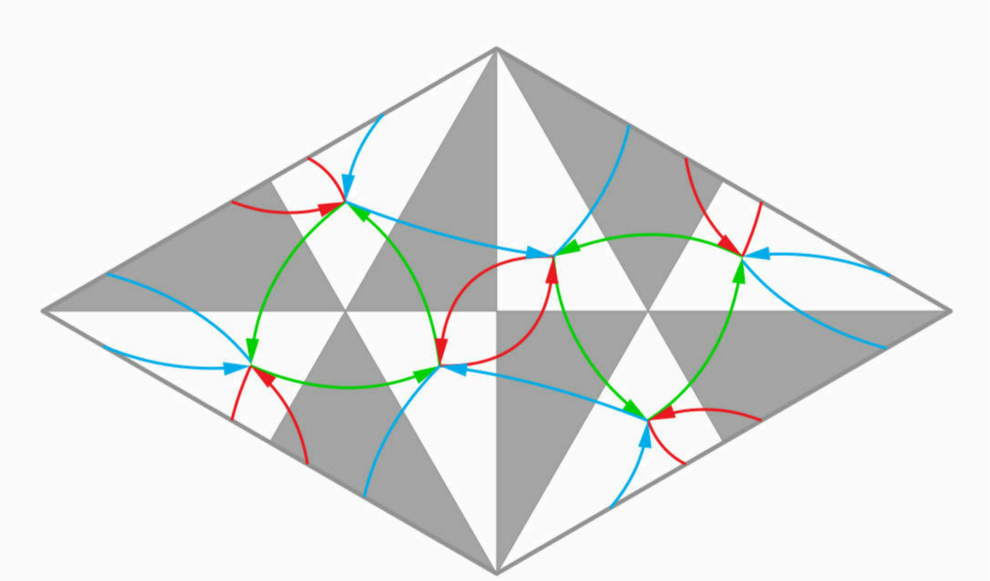

Figure 1: The first level Schreier graph of IMG (f_0).

2.2 Sketch of the proof

Step 2: Lift measure on S^2 to the solenoid $\mathcal{S}(f)$ with some special properties to make the adjacency graphs $\mathcal{G}_\infty$ of tiles on $\mathcal{S}(f)$ well-defined.

Step 3: Use the self-similarity of the solenoid $\mathcal{S}(f)$ to show that the $\mathcal{G}_n$ converges locally to $\mathcal{G}_\infty$.

Step 4: Apply the Benjamini–Schramm theorem to show that $\mathcal{G}_\infty$ is almost surely recurrent.

3. Future Work

- Simplify the proof of the main theorem by finding a more natural way to translate the Schreier graphs.
- Use the language in [3] to put the results in a more general context.
- Try to turn the almost surely recurrent property into a stronger statement, such as recurrent.
- **Prove the amenability of the iterated monodromy group of expanding Thurston maps**.

Bibliography

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